COMP 550
Algorithm and Analysis

Greedy Algorithms (Huffman Encoding)

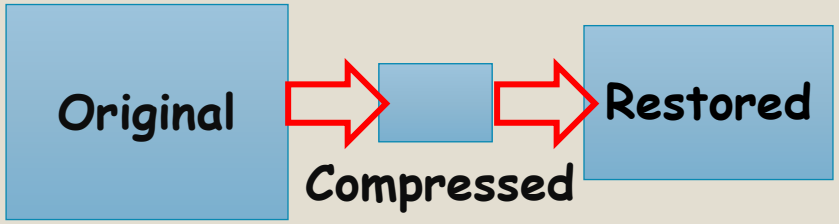
Based on CLRS Sec. 15.3

Some slides are adapted from ones by prior instructors Prof. Plaisted and Prof. Osborne

Data Compression

- Lossy

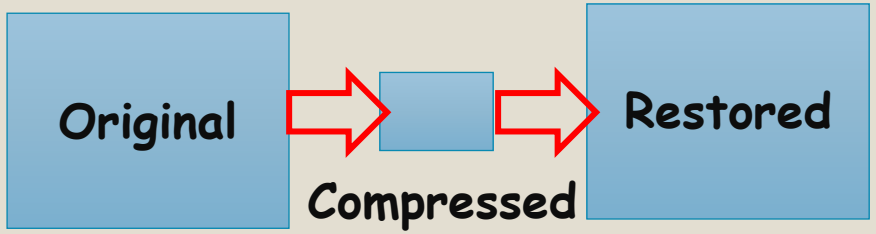
Big file →Small file → Imperfect copy



- Example: JPEG, MP3
- Idea: Targeted data deletion.
  - Delete parts of data that people won't notice.

- Lossless

Big file →Small file → Perfect copy



- Example: ZIP, PNG, JBIG.
- How to do this?

Fixed-Length Encoding

- Let's consider compression of text files
- How is text commonly represented?
  - Fixed length encoding (e.g., ASCII, Unicode)
  - 8 bits in ASCII
- Assuming ASCII, how many bits represent this text (ignore whitespace)?

she sells seashells by the seashore

  - Number of characters: 30
  - Bits required: $30 \times 8 = 240$

Fixed-Length Encoding

- Assuming ASCII, how many bits represent this text (ignore whitespace)?

she sells seashells by the seashore

  - Number of characters: 30
  - Bits required: $30 \times 8 = 240$

- There are 10 different characters in this text
  - At least 4 bits per character are required under fixed-length encoding scheme
  - Bits required: $30 \times 4 = 120$

Variable-Length Encoding

- Drawbacks of fixed-length codes
  - Wasted space
  - Same number of bits used to represent all characters
- **Potential solution**: use variable-length codes
  - Variable number of bits when frequency of occurrence is known
  - Short codes for characters that occur frequently

English letter frequencies (Wikipedia)

Letter	%
E	13
T	9.1
A	8.2
O	7.5
I	7
N	6.7
all others	

Variable-Length Encoding

she sells seashells by the seashore

Letter	Freq	Bit code	Bit count	Bits required
E	7	10	2	7x2=14
T	1	00	2	1x2=2
A	2	011	3	2x3=6
O	1	110	3	1x3=3
S	8	0101	4	8x4=32
H	4	0100	4	4x4=16
L	4	11100	5	4x5=20
Y	1	11101	5	1x5=5
B	1	11110	5	1x5=5
R	1	11111	5	1x5=5

Letter	%
E	13
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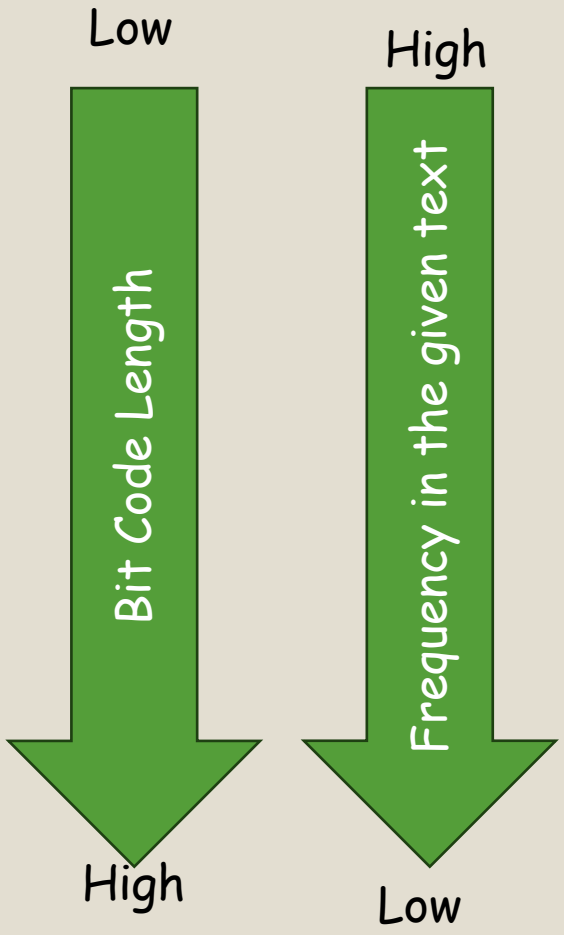
all others	

Number of bits = 108

Variable-Length Encoding

she sells seashells by the seashore



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L	4	110	3	4x3=12
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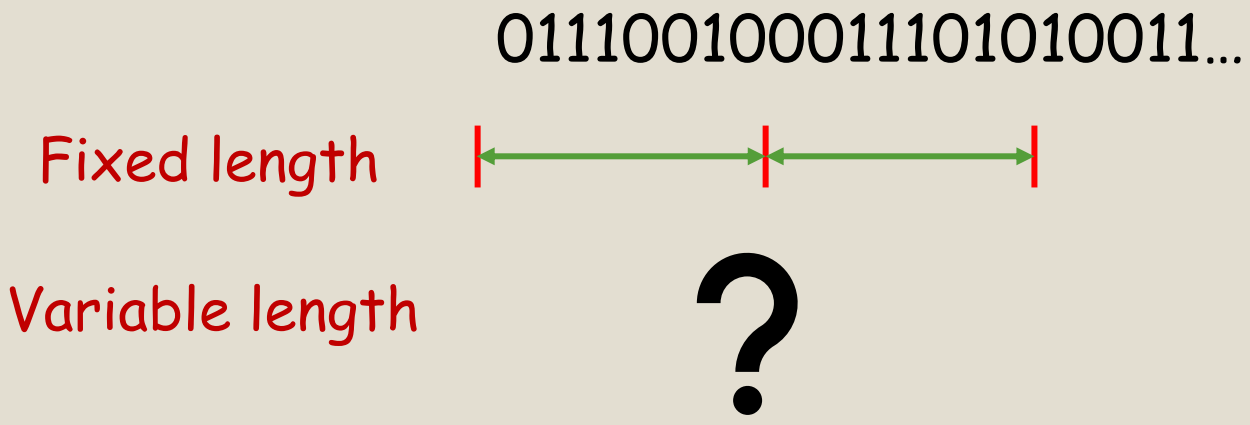
R	1	0100	4	1x4=4
B	1	11100	5	1x5=5
Y	1	11101	5	1x5=5
T	1	11110	5	1x5=5
O	1	11111	5	1x5=5

Number of bits = 86

Variable-Length Encoding

- **One issue:** Where a character ends and another begins?
  - Not a problem under fixed-length encoding



Prefix-Free Codes

- No codeword is a prefix of another codeword

No other codeword starts with 10

Letter	Freq	Bit code	Bit count	Bits required
S	8	10	2	8x2=16
E	7	00	2	7x2=14
H	4	011	3	4x3=12
L	4	110	3	4x3=12
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O	1	11111	5	1x5=5

Code: 0110011011011111

HELLO

Prefix-Free Codes

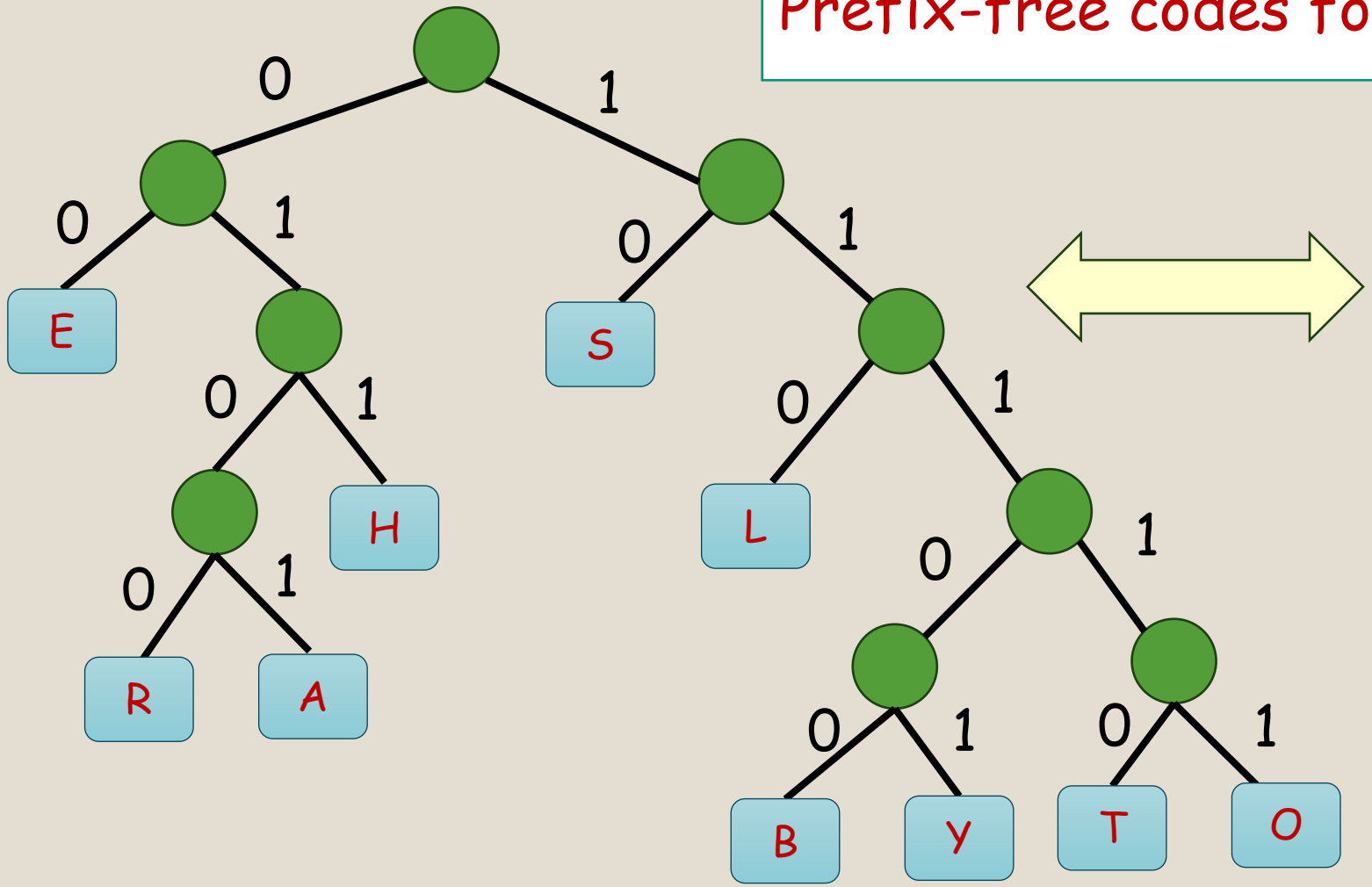
- The following codes do not satisfy prefix-free property

Letter	Bit code
S	0
E	1
H	01
L	10

- Decode the code 1010
  - Can be anyone among "ESES, LL, LES, ESL"

Prefix-Free Codes

Prefix-free codes form a binary tree



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Optimal Encoding

- Problem: Determine optimal prefix-free codes for a given text
  - Optimal = Minimize the number of bits to store the text
  - Minimize: $\sum$ frequecy of a char $\times$ Number of bits for the char
- Key property:
  - Less frequent characters are further down the tree

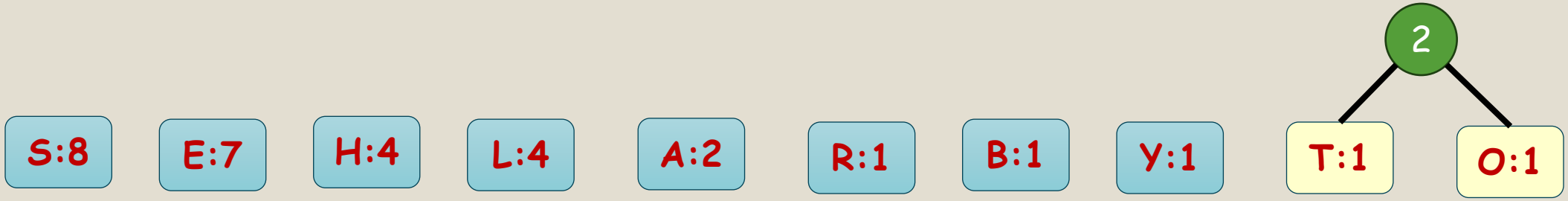
Huffman Encoding

1. Tabulate character frequencies

2. Each **char:frequency** tuple forms a single-node tree

S:8 E:7 H:4 L:4 A:2 R:1 B:1 Y:1 T:1 O:1

3. Merge the two trees with the smallest roots. The new root is the sum of the two frequencies below it. Repeat until there's one tree.



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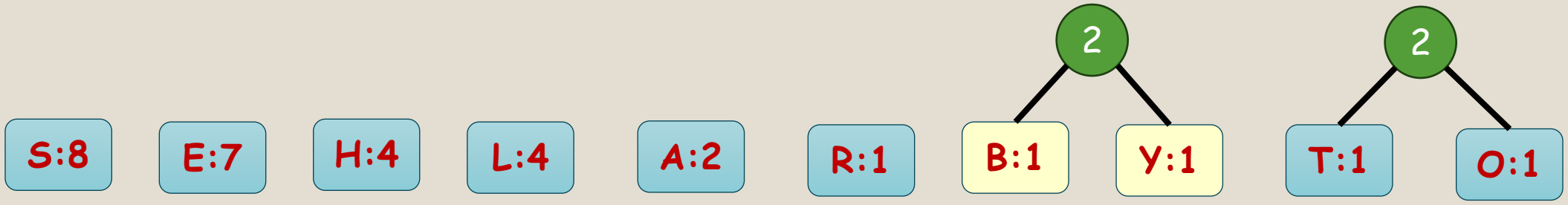

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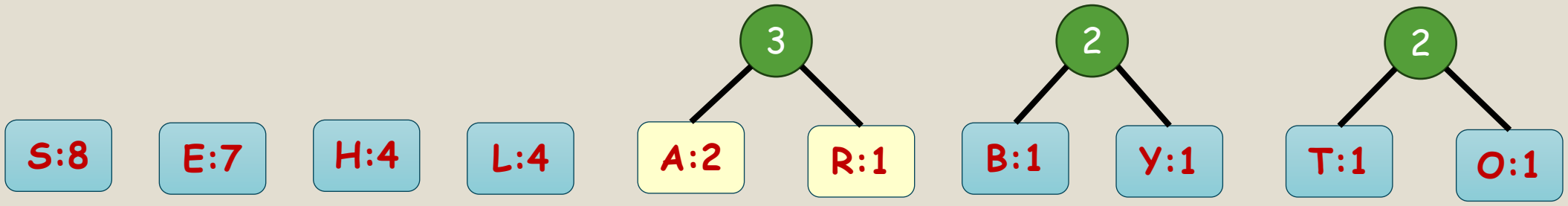

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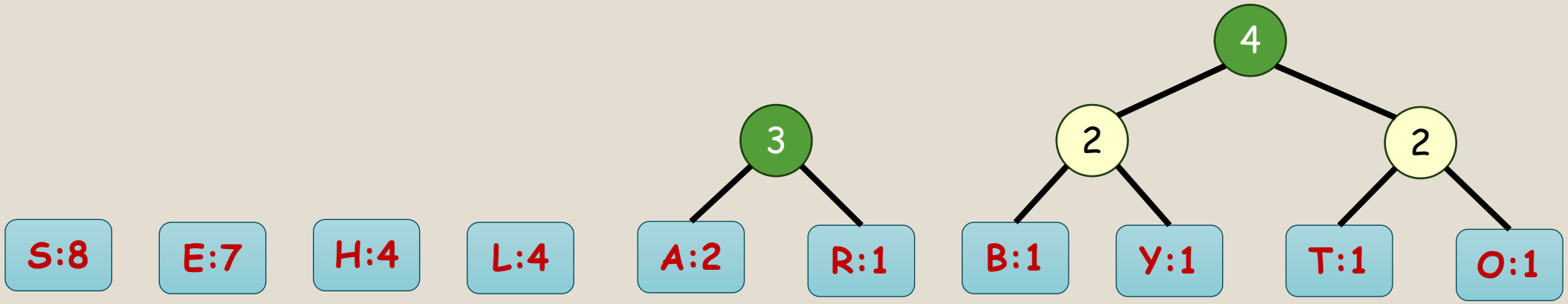

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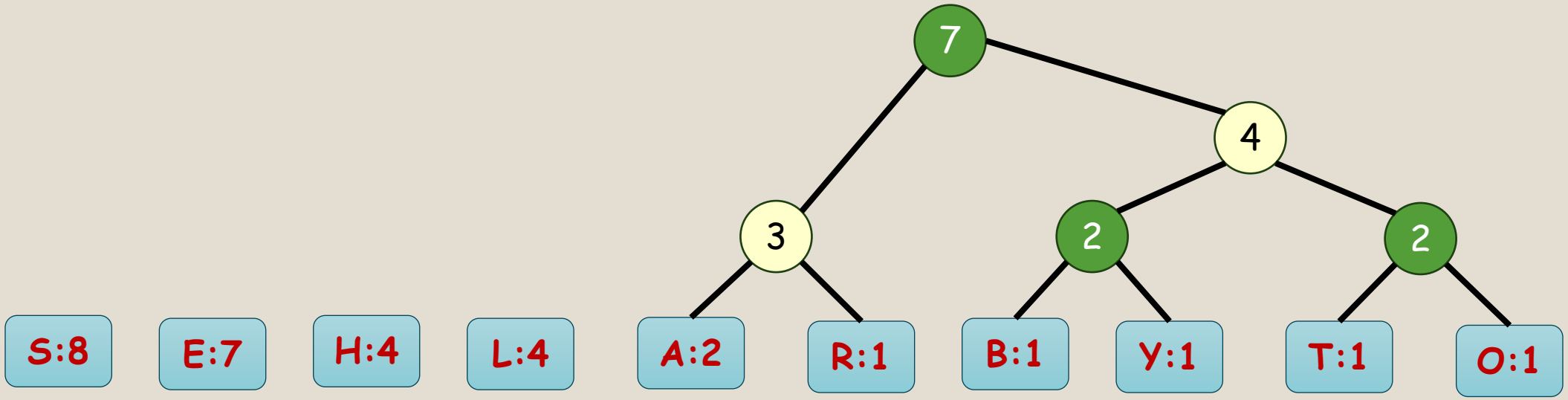

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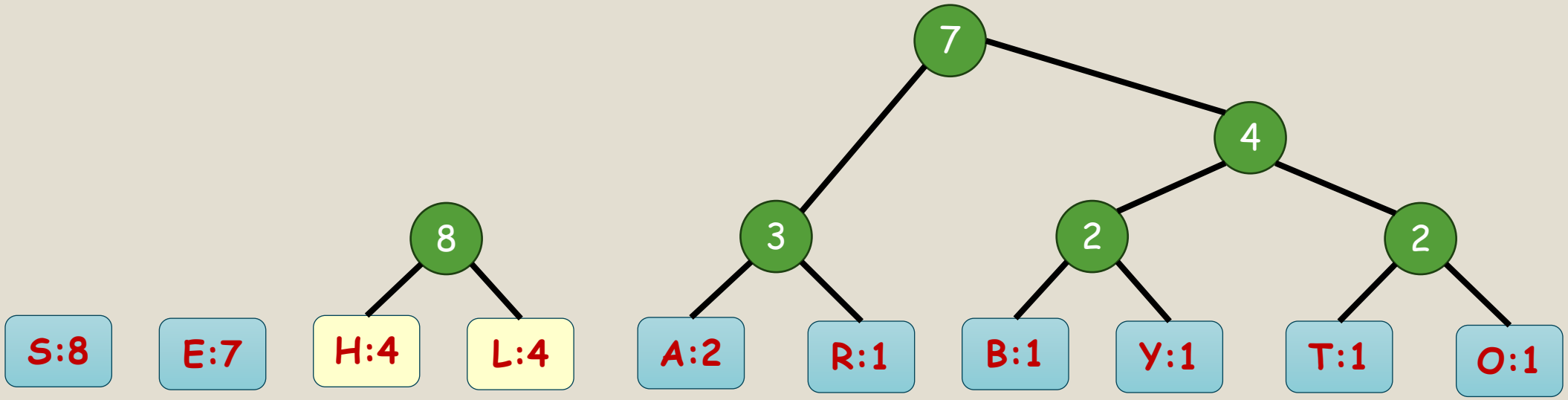

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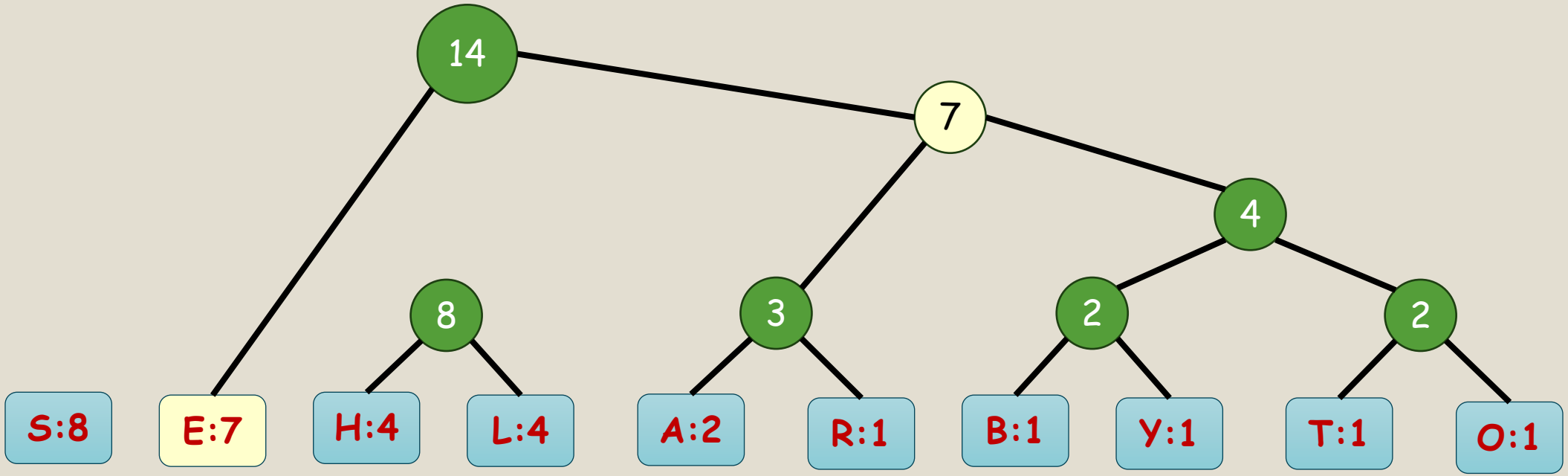

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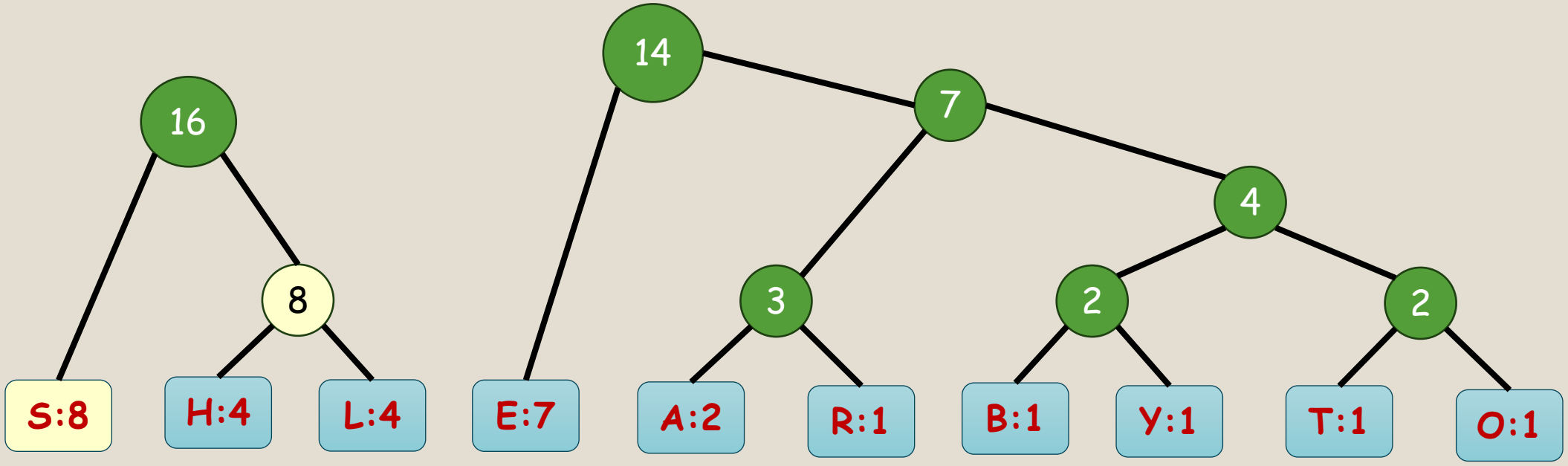

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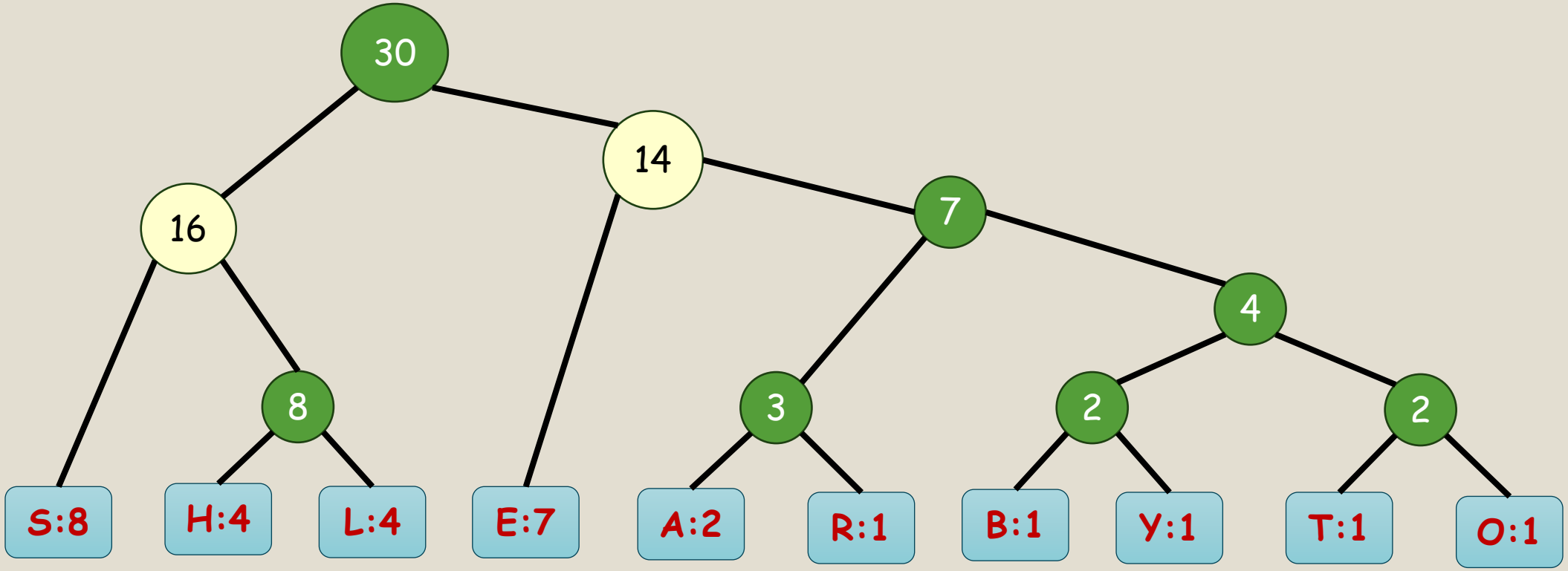

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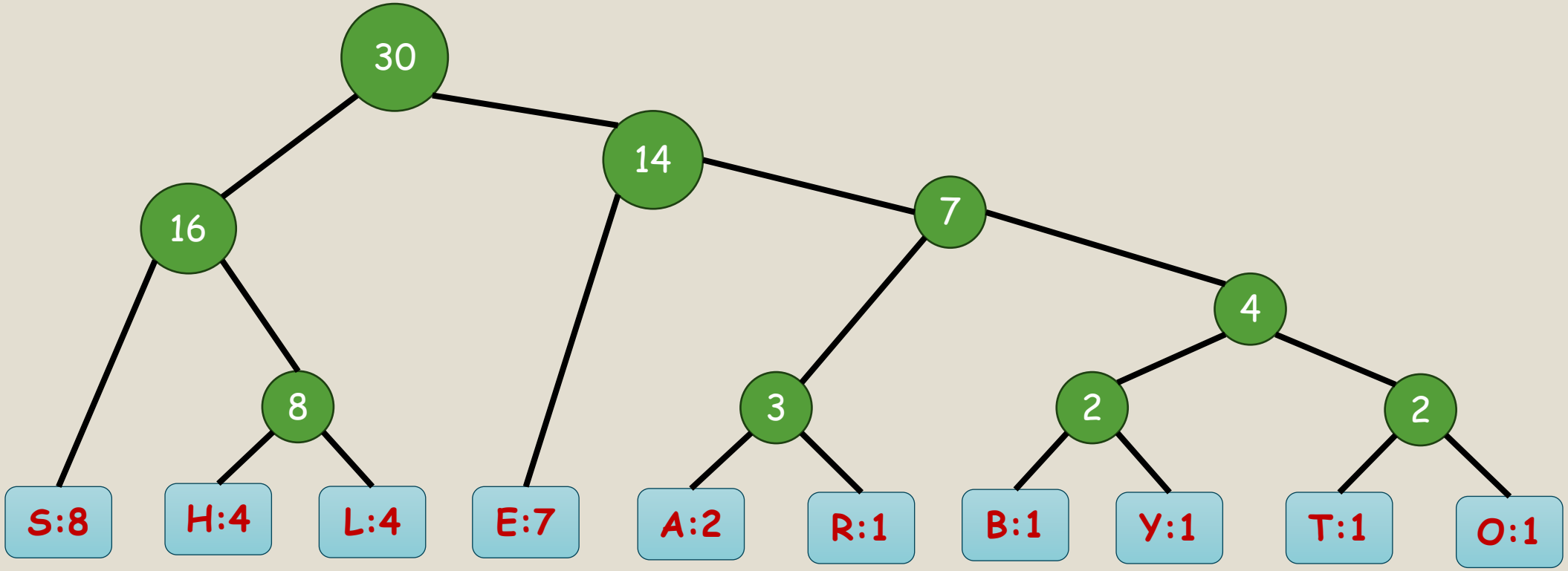
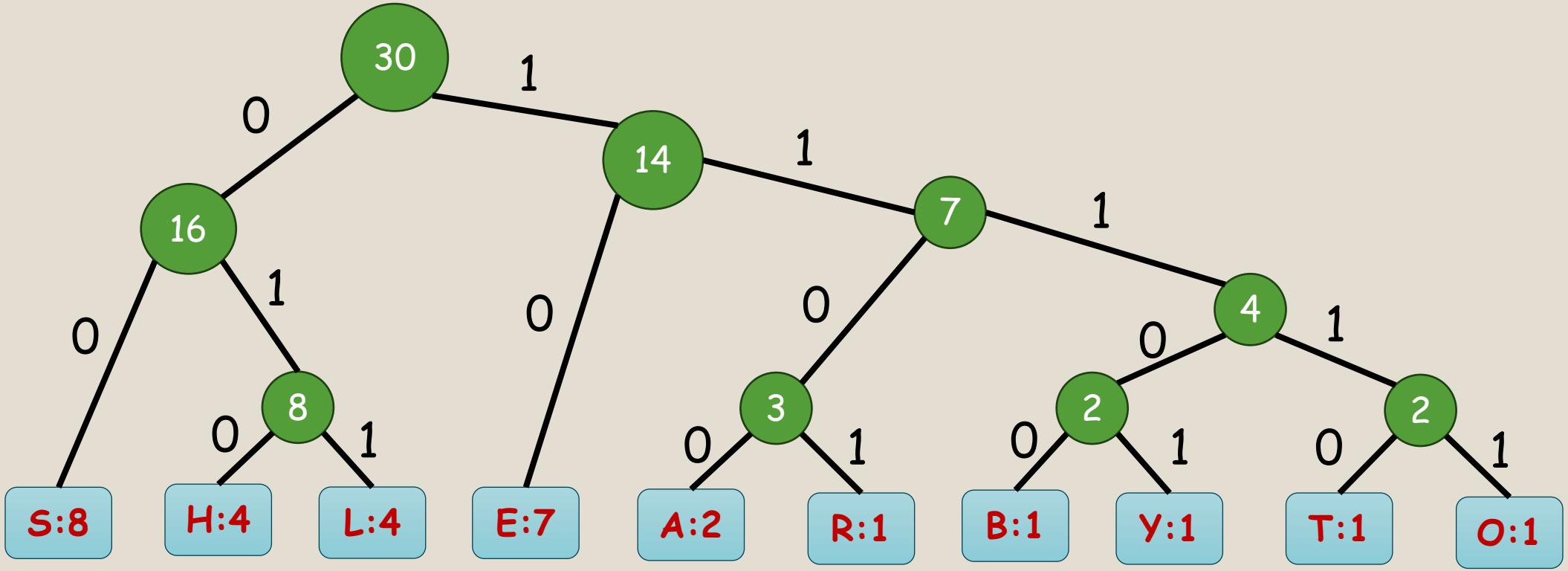


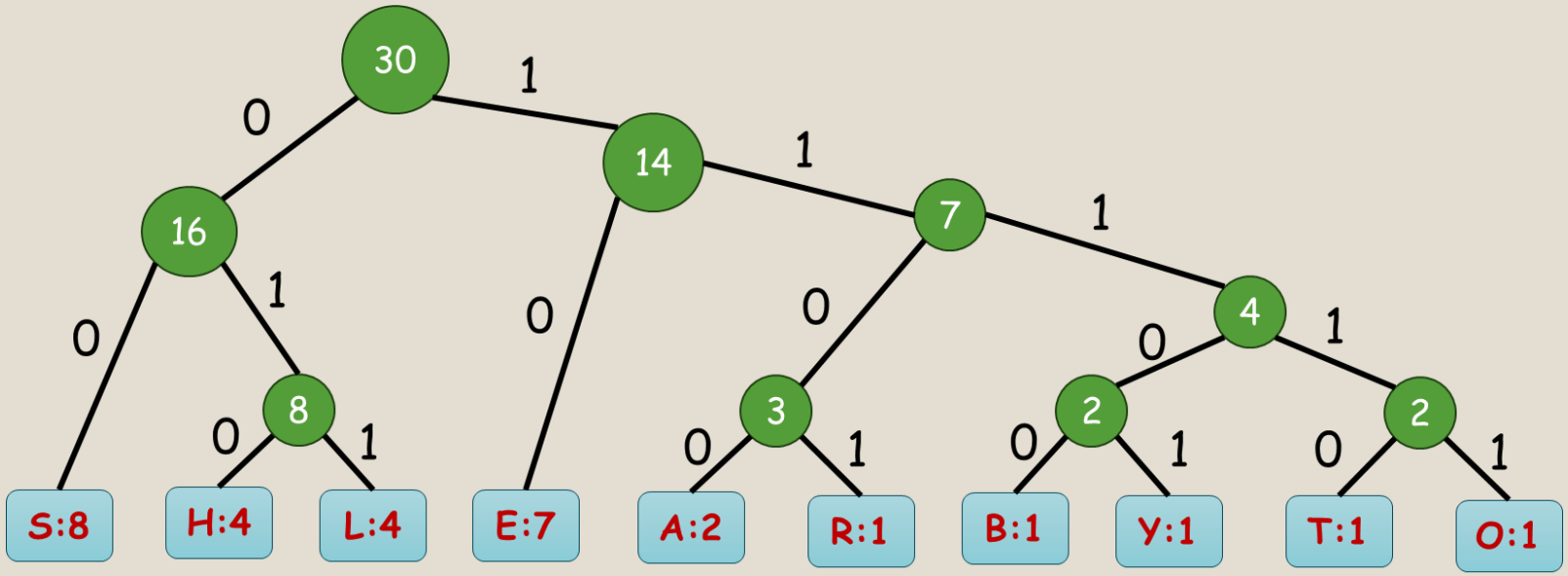
Huffman Encoding

4. Label left edges "0" and right edges "1"



Huffman Encoding

Codes are given by the path from root to leaf



Letter	Bit code
S	00
E	10
H	010
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A	1100
R	1101
B	11100
Y	11101
T	11110
O	11111

Huffman Encoding

- Input: A set C of n characters each with an attribute $freq$
- Output: A tree corresponding to optimal code

```
HUFFMAN(C)
 1 n = |C|
 2 Q = C
 3 for i = 1 to n − 1
 4     allocate a new node z
 5     x = EXTRACT-MIN(Q)
 6     y = EXTRACT-MIN(Q)
 7     z.left = x
 8     z.right = y
 9     z.freq = x.freq + y.freq
10     INSERT(Q, z)
11 return EXTRACT-MIN(Q)  // the root of the tree is the only node left
```

$O(\lg n)$ when we use MIN-Priority Queue

Running time $O(n \lg n)$

Correctness

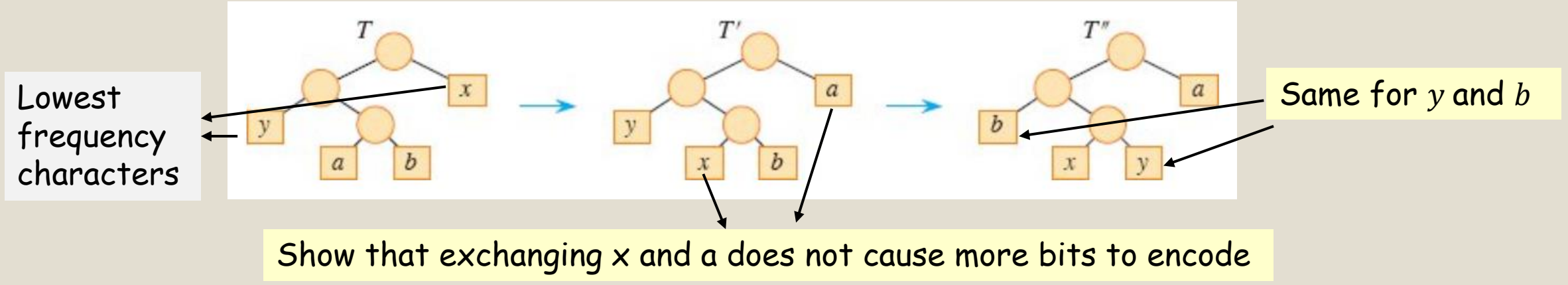
- To show that Huffman encoding builds an optimal prefix-free code:
  - The greedy choice of merging two trees with the smallest root satisfies the greedy-choice property
  - Constructing optimal prefix-free codes has the optimal substructure property

Greedy-Choice Property

There is an optimal solution containing the greedy choice (lowest frequency characters are siblings at the maximum depth)

Proof (sketch) by exchange argument:

- Take a tree T representing an arbitrary optimal prefix-free code
- Transform the tree T to a tree T' that has the greedy choice made by Huffman algorithm
- Show that T' is no worse than T



Optimal Substructure Property

Making the least frequency characters siblings and solving the new subproblem results in an optimal prefix-free code.

Goal: Replace the least frequency character x and y and a new node z with

$$z.freq = x.freq + y.freq$$

Optimal solution of problem = optimal solution of subproblem + (x and y as children of z)

Proof sketch: Assume that

Optimal solution of problem $\neq$ optimal solution of subproblem + (x and y as children of z)

- By the greedy-choice property, we know there is an optimal solution OPT where x and y are siblings.
- Removing x and y from OPT and replacing their parents in OPT by z results in a better solution for the subproblem, contradiction

Encoding & Decoding

- Encoding:
  - Create a hashmap or dictionary or array (indexed by characters) to store bit code of each character
  - Scan each character in the text, look for its code, and concatenate
- Decoding:
  - Go left/right the tree representing prefix-free code according to the current bit
  - If leaf is reached, concatenate that character to current text

Thank You!